A DSGE MODEL OF INSIDERS – OUTSIDERS CAPITALISM

MATHEMATICAL APPENDIX

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Section I. Notation

y_t: output of representative final good producer in period t

Y_t^i: composite input of intermediate goods produced in the insiders' sector

Y_t^o: composite input of intermediate goods produced in the outsiders' sector

A_t: Total factor productivity in final good sector

χ_t: share of insiders' industries in the economy

$\frac{1}{1-\phi}$: input elasticity of substitution across sectors

Composite inputs are defined by the Dixit-Stiglitz aggregator functions:

$$\mathrm{Y}_t^i = \left[\int_0^{\chi_t} y_t^i(\varsigma)^\theta d\varsigma\right]^{(1/\theta)} ;\theta \in (0,1) \quad \text{and} \quad \mathrm{Y}_t^o = \left[\int_{\chi_t}^1 y_t^o(\varsigma)^\theta d\varsigma\right]^{(1/\theta)}$$

$y_t^i(\varsigma)$: input of the intermediate good of the ς industry in insiders' sector

$y_t^o(\varsigma)$: input of the intermediate good of the ς industry in outsiders' sector

$\frac{1}{1-\theta}$: aggregation elasticity of substitution across industries

$p_t^i(\varsigma)$: real price of the intermediate good of the ς industry in the insiders' sector

$y_t^o(\varsigma)$: real price of the intermediate good of the ς industry in the outsiders' sector

$k_t^o(\varsigma)$: capital input

$l_t^o(\varsigma)$: labor input

B_t^o: total factor productivity in outsiders' industries

α: capital input elasticity

r_t: real rental cost of capital

w_t^o: real wage rate of outsiders

B_t^i: total factor productivity in insiders' industries

$w_t^i(\varsigma)$: real wage rate in the ς industry of the insiders' sector

Union preferences are characterized by a utility function of the form:

$$u_t^i(\varsigma) = \left[w_t^i(\varsigma) - w_t^o\right]^\lambda \left[l_t^i(\varsigma)\right]^{(1-\lambda)} ;\lambda \in (0,1)$$

λ: preference intensity of wage premium over union membership / employment

$$\left(w_t^i(\varsigma)^*, l_t^i(\varsigma)^*\right) = \underbrace{argmax}_{\left(w_t^i(\varsigma), l_t^i(\varsigma)\right)} \left\{\left[u_t^i(\varsigma) - \bar{u}_t^i(\varsigma)\right]^\mu \left[\pi_t^i(\varsigma) - \bar{\pi}_t^i(\varsigma)\right]^{1-\mu}\right\}; \mu \in (0,1)$$

$\bar{u}_t^i(\varsigma)$: union's reservation option

$\bar{\pi}_t^i(\varsigma)$: producer's reservation option

μ: union's relative bargaining power

c_t: (final good) consumption

k_t: capital stock at the beginning of period t

$\chi_t h_t^i$: labor supply to the insiders' sector

$(1-\chi_t)h_t^o$: labor supply to the outsiders' sector

$\chi_t \pi_t^i$: dividends from outsiders' industries

τ_t^K: capital income tax rate

τ_t^L: labor income tax rate

δ: fixed (geometric) capital depreciation rate

h_t: fraction of available household time devoted to work

$1-h_t$: fraction of available household time devoted to leisure

$$u_t^h = \frac{[c_t^{\vartheta}(1-h_t)^{1-\vartheta}]^{1-\gamma}}{1-\gamma}; \gamma > 0$$

$\beta \in (0,1)$ is the efficient household constant discount factor

$1/\gamma$ is the intertemporal elasticity of substitution

ϑ is the preference intensity for consumption

Parameter Restrictions

[R1] $$\lambda < \frac{(1-\alpha\theta)}{(1-\alpha\theta)+(1-\theta)} + \frac{(1-a)\theta}{(1-\alpha\theta)+(1-\theta)}\left(\frac{1-\mu}{\mu}\right)$$

[R2] $$\lambda > \frac{\alpha}{1+\alpha} - \frac{1-a}{1+\alpha}\left(\frac{1-\mu}{\mu}\right)$$

[R3] $$\frac{1-\upsilon}{1-\frac{\alpha\xi+(1-a)\theta}{\theta\xi}\upsilon} < \phi < \frac{1-\upsilon}{[\alpha\theta+(1-a)\xi]-\upsilon}$$

where $\upsilon \equiv \left(\theta^{\alpha}\xi^{(1-a)}\right)^{-\phi/(1-\phi)}$

SECTION II.

Proof of Proposition 1:

Given the strict concavity of the objective function in the second stage of the game, given by (I.17), the necessary and sufficient condition is (I.18). This condition gives the unique level of capital input:

$$k_t^i(\varsigma) = \left(\frac{\alpha\theta B_t^{i\theta} \Gamma_t^i}{r_t} \right)^{1/(1-\alpha\theta)} l_t^i(\varsigma)^{(1-\alpha)\theta/(1-a\theta)} \tag{A.1}$$

Using (A.1) to substitute out capital input in the producer's profit function (I.14), we can re-express the producer's profit function in the first stage of the game as:

$$\pi_t^i(\varsigma) = (1-\alpha\theta)\left[\frac{(\alpha\theta)^{\alpha\theta} B_t^{i\theta} \Gamma_t^i}{r_t^{\alpha\theta}} \right]^{1/(1-\alpha\theta)} l_t^i(\varsigma)^{(1-\alpha)\theta/(1-a\theta)} - w_t^i(\varsigma) l_t^i(\varsigma) \tag{A.2}$$

Therefore, the problem associated with the solution to the first stage of the game can be expressed as follows:

$$\max_{\left(w_t^i(\varsigma), l_t^i(\varsigma)\right)} \left(\left[w_t^i(\varsigma) - w_t^o \right]^{\lambda\left(\frac{\mu}{1-\mu}\right)} \left\{ (1-\alpha\theta)\left[\frac{(\alpha\theta)^{\alpha\theta} B_t^{i\theta} \Gamma_t^i}{r_t^{\alpha\theta}} \right]^{\left(\frac{1}{1-\alpha\theta}\right)} l_t^i(\varsigma)^{\left[\frac{(1-\alpha)\theta}{1-\alpha\theta} + (1-\lambda)\left(\frac{\mu}{1-\mu}\right)\right]} - w_t^i(\varsigma) l_t^i(\varsigma)^{1+(1-\lambda)\left(\frac{\mu}{1-\mu}\right)} \right\} \right) \tag{A.3}$$

subject to the union and producer participation constraints:

$$w_t^i(\varsigma) > w_t^o \tag{A.4}$$

$$(1-\alpha\theta)\left[\frac{(\alpha\theta)^{\alpha\theta} B_t^{i\theta} \Gamma_t^i}{r_t^{\alpha\theta}} \right]^{1/(1-\alpha)} l_t^i(\varsigma)^{(1-\alpha)\theta/(1-a\theta)} > w_t^i(\varsigma) l_t^i(\varsigma) \tag{A.5}$$

To conserve notation, this problem can be re-stated as follows:

$$\max_{(x_1, x_2) > \left(A_1, \left(\frac{A_2}{x_1}\right)^{1/(\alpha_3 - \alpha_2)} \right)} f(x_1, x_2) \tag{A.6}$$

where: $f : (A_1, \infty) x (0, \infty) \to \mathbb{R}$, $f(x_1, x_2) = (x_1 - A_1)^{\alpha_1} (A_2 x_2^{\alpha_2} - x_1 x_2^{\alpha_3})$, $x_1 = w_t^i(\varsigma)$, $x_2 = l_t^i(\varsigma)$, $A_1 = w_t^o$, $A_2 = (1-\alpha\theta)\left[\frac{(\alpha\theta)^{\alpha\theta} B_t^{i\theta} \Gamma_t^i}{r_t^{\alpha\theta}} \right]^{1/(1-\alpha\theta)}$, $\alpha_1 = \lambda\left(\frac{\mu}{1-\mu}\right)$, $\alpha_2 = \frac{(1-\alpha)\theta}{1-\alpha\theta} + (1-\lambda)\left(\frac{\mu}{1-\mu}\right)$, $\alpha_3 = 1 + (1-\lambda)\left(\frac{\mu}{1-\mu}\right)$.

Clearly, $A_1, A_2, \alpha_1, \alpha_2, \alpha_3 > 0$ and $\alpha_3 > \alpha_2$. Now, ignoring temporarily the constraints in (x_1, x_2), since f is differentiable and its domain is an open set, any extreme points of f satisfy the conditions : $f_{x_1} = f_{x_2} = 0$. That is,

$$\alpha_1(A_2 x_2^{\alpha_2} - x_1 x_2^{\alpha_3}) = (x_1 - A_1)x_2^{\alpha_3} \tag{A.7}$$

$$\alpha_2 A_2 x_2^{\alpha_2 - 1} = \alpha_3 x_1 x_2^{\alpha_3 - 1} \tag{A.8}$$

But, the above two-equation system in x_1 and x_2 has the unique solution:

$$x_1^* = \frac{1}{1 - \alpha_1\left(\frac{\alpha_3}{\alpha_2} - 1\right)} A_1 \tag{A.9}$$

$$x_2^* = \left(\frac{\alpha_2 A_2}{\alpha_3 x_1^*}\right)^{1/(\alpha_3 - \alpha_2)} \tag{A.10}$$

Hence, (x_1^*, x_2^*) is the solution to the problem defined by (A.6), provided that: (i) satisfies the participation constraints $(x_1, x_2) > \left(A_1, \left(\frac{A_2}{x_1}\right)^{1/(\alpha_3 - \alpha_2)}\right)$ and (ii) is a local maximum. Next, we show that given **[R1]**, both of these conditions are satisfied.

To show (i) note that $x_1^* > A_1$ if and only if $\alpha_1\left(\frac{\alpha_3}{\alpha_2} - 1\right) < 1$ and $\alpha_1\left(\frac{\alpha_3}{\alpha_2} - 1\right) < 1$ if and only if $\alpha_1 < \left(\frac{\alpha_2}{\alpha_3 - \alpha_2}\right)$ or $\lambda\left(\frac{\mu}{1-\mu}\right) < \left[\frac{\frac{(1-\alpha)\theta}{(1-a\theta)} + (1-\lambda)\left(\frac{\mu}{1-\mu}\right)}{\frac{(1-\theta)}{(1-a\theta)}}\right]$ or if and only if **[R1]** holds. Also, since $\alpha_3 > \alpha_2$ it is immediate from (A.10), that $x_2^* > \left(\frac{A_2}{x_1^*}\right)^{1/(a_3 - a_2)}$.

To show (ii) it suffices to show that f is strictly concave around (x_1^*, x_2^*), or since *f* is twice differentiable that $f_{x_1 x_1}^*, f_{x_2 x_2}^* < 0$ and $f_{x_1 x_1}^* f_{x_2 x_2}^* - \left(f_{x_1 x_2}^*\right)^2 > 0$. Using conditions (A.7) and (A.8), that are by construction satisfied by (x_1^*, x_2^*), it is straightforward that:

$$f^*_{x_1x_1} = -(1+\alpha_1)\left(x_1^* - A_1\right)^{(a_1-1)}(x_2^*)^{\alpha_3} < 0$$

$$f^*_{x_2x_2} = -\alpha_2\left(\frac{\alpha_3}{\alpha_2} - 1\right)\left(x_1^* - A_1\right)^{a_1}(x_2^*)^{(\alpha_3-2)} < 0, \text{ as } \alpha_3 > \alpha_2,$$

and

$$f^*_{x_1x_1} f^*_{x_2x_2} - \left(f^*_{x_1x_2}\right)^2 =$$
$$= \left(x_1^* - A_1\right)^{(2a_1-1)}(x_2^*)^{(\alpha_3-2)}\{[(1+\alpha_1)[\alpha_3^2 x_1^*(x_2^*)^{\alpha_3} - \alpha_2^2 A_2(x_2^*)^{\alpha_2}] - \alpha_3^2(x_1^* - A_1)(x_2^*)^{\alpha_3}\}.$$

Therefore, $f^*_{x_1x_1} f^*_{x_2x_2} - \left(f^*_{x_1x_2}\right)^2 > 0$ if and only if:

$$[(1+\alpha_1)[\alpha_3^2 x_1^*(x_2^*)^{\alpha_3} - \alpha_2^2 A_2(x_2^*)^{\alpha_2}] - \alpha_3^2(x_1^* - A_1)(x_2^*)^{\alpha_3} > 0.$$

Using (A.9) to substitute for $(x_1^* - A_1)$ in the LHS of the above inequality, it follows that this inequality holds if and only if: $\frac{x_1^*(x_2^*)^{\alpha_3}}{A_2(x_2^*)^{\alpha_2}} > \frac{(1+\alpha_1)\alpha_2^2 + \alpha_1\alpha_3^2}{(1+\alpha_1)\alpha_3^2 + \alpha_1\alpha_3^2}$. But, (A.10) implies that $\frac{x_1^*(x_2^*)^{\alpha_3}}{A_2(x_2^*)^{\alpha_2}} = \frac{\alpha_2}{\alpha_3}$. Therefore, the above inequality holds if and only if:

$\frac{\alpha_2}{\alpha_3} > \frac{(1+\alpha_1)\alpha_2^2 + \alpha_1\alpha_3^2}{(1+\alpha_1)\alpha_3^2 + \alpha_1\alpha_3^2}$. But, then the last inequality holds if and only if:

$$\left(\frac{\alpha_2}{\alpha_3}\right)^2 - \left(1 + \frac{\alpha_1}{1+\alpha_1}\right)\left(\frac{\alpha_2}{\alpha_3}\right) + \left(\frac{\alpha_1}{1+\alpha_1}\right) < 0, \text{ or } \left[\left(\frac{\alpha_2}{\alpha_3}\right) - 1\right]\left[\left(\frac{\alpha_2}{\alpha_3}\right) - \left(\frac{\alpha_1}{1+\alpha_1}\right)\right] < 0.$$

Then, since $\alpha_3 > \alpha_2$ the last inequality holds if and only if $\left(\frac{\alpha_2}{\alpha_3}\right) > \left(\frac{\alpha_1}{1+\alpha_1}\right)$, or $\alpha_1\left(\frac{\alpha_3}{\alpha_2} - 1\right) < 1$, which, as already shown, is equivalent to restriction **[R1]**. Hence, given **[R1]**, $f^*_{x_1x_1} f^*_{x_2x_2} - \left(f^*_{x_1x_2}\right)^2 > 0$. And, therefore (x_1^*, x_2^*) is a local maximum of $f(x_1, x_2)$ It follows that given **[R1]**, (x_1^*, x_2^*) is the unique global maximum of $f(x_1, x_2)$ and satisfies the participation constraints, $(x_1, x_2) > \left(A_1, \left(\frac{A_2}{x_1}\right)^{1/(\alpha_3-\alpha_2)}\right)$. Hence, (x_1^*, x_2^*) is a unique solution to (A.6).

Now, returning to the original notation, observe that, in view of the definition of ν in (I.23), (A.9) implies (I.20). Further, (A.10) implies:

$$(1-\alpha\theta)\left[\frac{\frac{(1-\alpha)\theta}{1-\alpha\theta}+(1-\lambda)\left(\frac{\mu}{1-\mu}\right)}{1+(1-\lambda)\left(\frac{\mu}{1-\mu}\right)}\right]\left[\frac{(\alpha\theta)^{\alpha\theta}B_t^{i\theta}\Gamma_t^i}{r_t^{\alpha\theta}}\right]^{1/(1-\alpha\theta)} l_t^i(\varsigma)^{(1-\alpha)\theta/(1-a\theta)} = w_t^i(\varsigma)l_t^i(\varsigma)$$

The latter, in view of (A.2) and (I.18), implies that:

$$(1-\alpha\theta)\left[\frac{\frac{(1-\alpha)\theta}{1-\alpha\theta}+(1-\lambda)\left(\frac{\mu}{1-\mu}\right)}{1+(1-\lambda)\left(\frac{\mu}{1-\mu}\right)}\right] p_t^i(\varsigma)y_t^i(\varsigma) = w_t^i(\varsigma)l_t^i(\varsigma) \tag{A.11}$$

But, in view of the definition in view of the definitions of ν in (I.23) and ξ in (I.24),

$$(1-\alpha\theta)\left[\frac{\frac{(1-\alpha)\theta}{1-\alpha\theta}+(1-\lambda)\left(\frac{\mu}{1-\mu}\right)}{1+(1-\lambda)\left(\frac{\mu}{1-\mu}\right)}\right] = (1-\alpha)\nu\xi \tag{A.12}$$

Therefore, (A.11) and (A.12) imply (I.19). Finally, in view of the definition of κ in (I.22), (I.21) follows from the definition of profits in (I.14) and (I.18) and (I.19). Finally, to show that κ>0, note that in view of the definition of κ in (I.22) and (A.12),

$$\kappa = (1-\alpha\theta)\left(1-\frac{1-\alpha}{1-\alpha\theta}\nu\xi\right) = (1-\alpha\theta)\left[1-\frac{\frac{(1-\alpha)\theta}{1-\alpha\theta}+(1-\lambda)\left(\frac{\mu}{1-\mu}\right)}{1+(1-\lambda)\left(\frac{\mu}{1-\mu}\right)}\right]$$

And, since $\alpha,\theta\in(0,1)$, it is straightforward from the above equation that κ>0.

This completes the proof of Proposition 1.

SECTION III.

Proof of Proposition 2:

Part a: In view of symmetry characterizing the private sector equilibrium, industry demand functions (I.5) and (I.6) imply:

$$\left(\frac{p_t^i}{p_t^o}\right)=\left(\frac{\Gamma_t^i}{\Gamma_t^o}\right)\left(\frac{y_t^i}{y_t^o}\right)^{(\theta-1)} = \left(\frac{\chi_t}{1-\chi_t}\right)\left(\frac{Y_t^i}{Y_t^o}\right)^{(\phi-\theta)}\left(\frac{y_t^i}{y_t^o}\right)^{(\theta-1)}$$

From the sectoral aggregation functions (I.2) and (I.3), $Y_t^i=\chi_t^{1/\theta}y_t^i$ and $Y_t^o=(1-\chi_t)^{1/\theta}y_t^o$, respectively. Thus, we can re-express the above equation as:

$$\left(\frac{p_t^i}{p_t^o}\right)=\left(\frac{\chi_t}{1-\chi_t}\right)^{\phi/\theta}\left(\frac{y_t^i}{y_t^o}\right)^{(\phi-1)} \tag{A.13}$$

From the industry capital input demand equations (I.11) and (I.18):

$$\left(\frac{k_t^i}{k_t^o}\right)=\theta\left(\frac{p_t^i}{p_t^o}\right)\left(\frac{y_t^i}{y_t^o}\right) \text{ or, in view of (II.20), } \left(\frac{k_t^i}{k_t^o}\right)=\theta\left(\frac{\chi_t}{1-\chi_t}\right)^{\phi/\theta}\left(\frac{y_t^i}{y_t^o}\right)^{\phi}$$

Now, multiplying both hand sides of the above equation by $\frac{\chi_t}{1-\chi_t}$, we have:

$$\left[\frac{\chi_t k_t^i}{(1-\chi_t)k_t^o}\right]=\theta\left(\frac{\chi_t}{1-\chi_t}\right)^{1+\frac{\phi(1-\theta)}{\theta}}\left[\frac{\chi_t y_t^i}{(1-\chi_t)y_t^o}\right]^{\phi} \tag{A.14}$$

Likewise, from the industry labor input demand equations (I.12) and (I.18), in view of the insiders' wage premium equation (I.20) and the labor market equilibrium conditions (II.2) and (II.3), we have

$$\left[\frac{\chi_t h_t^i}{(1-\chi_t)h_t^o}\right]=\xi\left(\frac{\chi_t}{1-\chi_t}\right)^{1+\frac{\phi(1-\theta)}{\theta}}\left[\frac{\chi_t y_t^i}{(1-\chi_t)y_t^o}\right]^{\phi} \tag{A.15}$$

But, from the industry output supply functions (I.9) and (I.13) and in view of the fact that these functions are homogeneous of degree one, we have:

$$\left[\frac{\chi_t y_t^i}{(1-\chi_t)y_t^o}\right]=\left(\frac{B_t^i}{B_t^o}\right)\left[\frac{\chi_t k_t^i}{(1-\chi_t)k_t^o}\right]^{\alpha}\left[\frac{\chi_t h_t^i}{(1-\chi_t)h_t^o}\right]^{(1-a)}$$

The last equation, in view of (A.14) and (A.15), gives the important result:

$$\left[\frac{\chi_t y_t^i}{(1-\chi_t)y_t^o}\right]=\left[\theta^{\alpha}\xi^{(1-\alpha)}\left(\frac{B_t^i}{B_t^o}\right)\right]^{1/(1-\phi)}\left(\frac{\chi_t}{1-\chi_t}\right)^{1+\frac{\phi}{\theta(1-\phi)}} \tag{A.16}$$

And, in view of the definition of $\Delta_t(\chi_t)$ in (II.13), (A.16) yields (II.9). Now, in view of (II.9), (A.14) and (A.15) give (II.5) and (II.7), respectively. Moreover, (II.5) and the market equilibrium for capital (II.1) imply (II.6); and, (II.7) and the household time constraint (I.26) imply (II.8). Furthermore, the supply of output in the outsiders' industries (I.9), in view of (II.6) and (II.8) yields (II.10). Finally, the price ratio in (II.11) follows from (A.13) and (A.16).

Part b: Combining the industry output demand and the demand for capital input in the outsiders' sector, (I.6) and (I.11), respectively, we have:

$$r_t=\frac{\alpha(1-\chi_t)\tilde{A}_t^{\phi}y_t^{(1-\phi)}Y_t^{o(\phi-\theta)}y_t^{o\theta}}{k_t^o}=\alpha(1-\chi_t)A_t^{\phi}\left(\frac{Y_t^o}{y_t}\right)^{\phi}\left(\frac{y_t^o}{Y_t^o}\right)^{\phi}\left(\frac{y_t}{k_t^o}\right) \tag{A.17}$$

Consider the expression $A_t^{\phi}\left(\frac{Y_t^o}{y_t}\right)^{\phi}\left(\frac{y_t^o}{Y_t^o}\right)^{\phi}$. Using the aggregate output supply (I.1) and the industry aggregation function for the outsiders' sector (I.3), we have:

$$\begin{aligned}A_t^{\phi}\left(\frac{Y_t^o}{y_t}\right)^{\phi}\left(\frac{y_t^o}{Y_t^o}\right)^{\theta} &= \frac{1}{\left[1+\left(\frac{\chi_t}{1-\chi_t}\right)^{1+\frac{\phi}{\theta}}\left(\frac{y_t^i}{y_t^o}\right)^{\phi}\right](1-\chi_t)}\\ &= \frac{1}{\left[1+\left(\frac{\chi_t}{1-\chi_t}\right)^{1-\phi+\frac{\phi}{\theta}}\left(\frac{\chi_t y_t^i}{(1-\chi_t)y_t^o}\right)^{\phi}\right](1-\chi_t)}\\ &= \frac{1}{\left[1+\Delta_t(\chi_t)\right](1-\chi_t)}\end{aligned} \tag{A.18}$$

Where, the last equation follows from results established in the proof of Part a. Therefore, (A.17) and (A.18) imply:

$$r_t=\alpha\frac{1}{\left[1+\Delta_t(\chi_t)\right]}\frac{y_t}{(1-\chi_t)k_t^o} \tag{A.19}$$

Now, in view of (II.6) and the definition of $\hat{\omega}(x)$ in (II.18), (A.19) yields (II.14). Following a similar procedure, we can establish (II.15). To establish (II.16), we use the profits in the representative insiders' industry derived in Proposition 1 (i.e., (I.21)) to get: $\chi_t \pi_t^i = \kappa p_t^i \chi_t y_t^i$, which, in view of (I.19), (II.7), (II.8), and (II.15) leads to (II.16). Finally, the final good supply, (II.17), follows from the final good production function, (I.1), by a similar procedure as in the proof of (II.14).

Part c: Condition (II.21) follows from the household intra-temporal condition (I.31) and the wage equation (II.15). Condition (II.22) follows from the household Euler condition (I.32) and the real rental cost of capital equation (II.14). And, the government budget constraint equation (II.23) follows from (I.34) and (I.35), using results established in Parts a and b, above.

Finally, it remains to show that , given z_t and government policy variables $\left(\bar{\psi}_t, \tau_t^k, \chi_t\right)$, the laws of motion of the aggregate state of the private sector equilibrium depend only on the aggregate state of the private sector (k_t, h_t) and (ii) they are characterized, completely, in terms of: the transversality condition (I.33); the initial condition, $k_0 \in (0,\infty)$ given; the resource constraint (II.4) and conditions (II.21)-(II.22). This simply follows from the fact that (II.4) can be used to express household consumption, c_t, and the labour income tax rate, τ_t^L, in terms of the aggregate state (k_t, h_t). Hence, it follows that all variables used to define the private sector equilibrium can be expressed exclusively in terms of the aggregate state, the exogenous variables z_t and the policy variables $\left(\bar{\psi}_t, \tau_t^k, \chi_t\right)$.

This completes the proof of Proposition 2.

SECTION IV.

The Equations summarising the Politico-Economic Equilibrium

Economy-wide variables:

Aggregate production function:

$$y_t = \bar{\omega}_t(\chi_t) k_t^{\alpha} h_t^{(1-\alpha)}$$

Resource constraint:

$$y_t = c_t + [(1+\eta)k_{t+1} - (1-\delta)k_t] + g_t$$

Government budget constraint:

$$g_t = \left\{ \tau_t^K \left[\alpha \hat{\omega}_t(\chi_t) - \delta \left(\frac{k_t}{y_t} \right) \right] + \tau_t^L (1-\alpha) \tilde{\omega}_t(\chi_t) \right\} y_t$$

Intra-temporal condition:

$$\frac{1-\vartheta}{\vartheta} \frac{c_t}{1-h_t} = (1-\tau_t^L)(1-\alpha) \breve{\omega}_t(\chi_t) \frac{y_t}{h_t}$$

Euler condition for capital:

$$\frac{u_t}{c_t} = \frac{\beta}{(1+\eta)} E_t \left\{ \frac{u_{t+1}}{c_{t+1}} \left[1 + (1-\tau_{t+1}^K)[\hat{\omega}_{t+1}(\chi_{t+1}) \alpha \frac{y_{t+1}}{k_{t+1}} - \delta] \right] \right\}$$

Transversality condition:

$$\beta^T E_0 \frac{u_T k_{T+1}}{c_T} \to 0 \ \ as \ \ T \to \infty$$

Initial condition:

$$(k_0, \chi_0) \in (0, \infty) \times (0,1) \ \text{ given}$$

Prices and Dividends

Real Rental Cost of Capital:

$$r_t = \alpha \hat{\omega}_t(\chi_t) \frac{y_t}{k_t}$$

Dividends:

$$\chi_t w_t^i h_t^i + (1-\chi_t) w_t^o h_t^o + \chi_t \pi_t^i = (1-\alpha) \tilde{\omega}_t(\chi_t) y_t$$

sector wages

$$w_t^o = (1-\alpha)\breve{\omega}_t(\chi_t)\frac{y_t}{h_t}$$

$$w_t^i = (1-\alpha)\nu\breve{\omega}_t(\chi_t)\frac{y_t}{h_t}$$

Wedges and Auxiliary Variables

$$\hat{\omega}_t(\chi_t) = \frac{1+\theta\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)}$$

$$\breve{\omega}_t(\chi_t) = \frac{1+\xi\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)}$$

$$\bar{\omega}_t(\chi_t) = A_t B_t^o \frac{(1-\chi_t)^{\frac{\theta(1-\phi)+\phi}{\theta\phi}}[1+\Delta_t(\chi_t)]^{\frac{1}{\phi}}}{[1+\theta\Delta_t(\chi_t)]^{\alpha}[1+\xi\Delta_t(\chi_t)]^{(1-\alpha)}}$$

$$\tilde{\omega}_t(\chi_t) = \frac{1+o\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)}$$

$$\Delta_t^g(\chi_t,\chi_{t+1}) = \bar{\psi}_t + \hat{\psi}\chi_t + \frac{1}{2}\tilde{\psi}(\chi_{t+1}-\chi_t)^2$$

$$\Delta_t(\chi_t) = \left[\theta^{\alpha}\xi^{(1-a)}\left(\frac{B_t^i}{B_t^0}\right)\right]^{-\frac{\phi}{1-\phi}}\left(\frac{\chi_t}{1-\chi_t}\right)^{\frac{\theta(1-\phi)+\phi}{\theta(1-\phi)}}$$

where

$$\nu = \frac{1}{1-\dfrac{(1-\theta)\lambda}{(1-\alpha\theta)(1-\lambda)+(1-\alpha)\theta\left(\dfrac{1-\mu}{\mu}\right)}} > 1$$

$$\xi = \frac{(1-\alpha)\theta + \{1-\alpha\theta-[(1-\alpha\theta)+(1-\theta)]\lambda\}\left(\dfrac{\mu}{1-\mu}\right)}{(1-\alpha)\left[1+(1-\lambda)\left(\dfrac{\mu}{1-\mu}\right)\right]} < 1$$

$$o = \frac{1-\alpha\theta}{1-\alpha} > 1$$

SECTION V.

Optimal Policy of Hybrid Government: Primal Approach

In this case, the equations characterizing the PSE which constrain the government's optimal decision, can be summarized in the so called implementability constraint. For the methodology, see for example, Chari et al. (1994) and Ljungqvist and Sargent (2018, Chapter 16).

A. The Implementability Constraint and the Government Spending Policy Augmented Resource Constraint

The equations characterizing the economy-wide variables of the PSE, (III.1) -(III.7), are equivalent to the following three conditions:

(i) The "Implementability Constraint" (IC):

$$E_0\sum_{t=0}^{\infty}\beta^t\tilde{\varsigma}\,(h_t,\chi_t;z_t)u^h(c_t,h_t)=\hat{\varsigma}(c_0,h_0,k_0,\chi_0;z_0)u^h(c_o,h_0) \tag{H.1}$$

where:

$$\hat{\varsigma}(c_0,h_0,k_0,\chi_0;z_0)\equiv\frac{\alpha(1-\tau_0^K)\hat{\omega}_0(\chi_0)\bar{\omega}_0(\chi_0)k_0^{\alpha}h_0^{1-\alpha}+\left[1-(1-\tau_0^K)\delta\right]k_0}{c_0} \tag{H.2}$$

$$\tilde{\varsigma}\,(h_t,\chi_t;z_t)\equiv 1-\varpi_t(\chi_t)\frac{(1-\vartheta)h_t}{\vartheta(1-h_t)} \tag{H.3}$$

and, given parameter restrictions [R1] - [R2]:

$$\varpi_t(\chi_t)\equiv\frac{\tilde{\omega}_t(\chi_t)}{\bar{\omega}_t(\chi_t)}=\begin{cases}1, & \text{in the prototype economy with } \chi_t=0,\forall t\in\mathbb{N}_+\\ 1, & \text{in the prototype economy with } \chi_t\in(0,1),\forall t\in\mathbb{N}_+\\ \dfrac{1+o\Delta_t(\chi_t)}{1+\xi\Delta_t(\chi_t)}>1, & \text{in the detailed economy with } \chi_t\in(0,1),\forall t\in\mathbb{N}_+\\ o>1, & \text{in the detailed economy with } \chi_t=1,\forall t\in\mathbb{N}_+\end{cases} \tag{H.4}$$

(ii) The "government spending policy augmented resource constraint" (GSPARC):

$$c_t=\left[1-\Delta_t^g(\chi_t,\chi_{t+1})\right]\bar{\omega}_t(\chi_t)k_t^{\alpha}h_t^{(1-\alpha)}-\left[(1+\eta)k_{t+1}-(1-\delta)k_t\right] \tag{H.5}$$

where, the government spending policy function $\Delta_t^g(\chi_t,\chi_{t+1})$ is defined in (III.3)

(iii) The initial condition, (III.7).

Proof

First, note that combining the resource constraint (III.2) and the government budget constraint (III.3), we have:

$$y_t = c_t + (1+\eta)k_{t+1} - (1-\delta)k_t + \left\{\tau_t^K\left[\alpha\hat{\omega}_t(\chi_t) - \delta\left(\frac{k_t}{y_t}\right)\right] + \tau_t^L(1-\alpha)\tilde{\omega}_t(\chi_t)\right\}y_t \qquad \text{(A.1)}$$

Second, note that:

$$\begin{aligned}\alpha\hat{\omega}_t(\chi_t) + (1-\alpha)\tilde{\omega}_t(\chi_t) &= \alpha\frac{1+\theta\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)} + (1-\alpha)\frac{1+o\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)} \\ &= \frac{\alpha+\alpha\theta\Delta_t(\chi_t)] + 1 - \alpha + (1-\alpha\theta)\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)} \\ &= \frac{1+\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)} \\ &= 1\end{aligned}$$

Hence, (A1) can be re-written as:

$$c_t - (1-\tau_t^L)(1-\alpha)\tilde{\omega}_t(\chi_t)y_t = -(1+\eta)k_{t+1} + \left\{1 + (1-\tau_t^K)\left[\alpha\hat{\omega}_t(\chi_t)\left(\frac{y_t}{k_t}\right) - \delta\right]\right\}k_t$$

Further, in view of the intra-temporal condition (III. 4), the preceding equation yields:

$$c_t\left[1 - \varpi_t(\chi_t)\frac{(1-\vartheta)h_t}{\vartheta(1-h_t)}\right] = -(1+\eta)k_{t+1} + \left\{1 + (1-\tau_t^K)\left[\alpha\hat{\omega}_t(\chi_t)\left(\frac{y_t}{k_t}\right) - \delta\right]\right\} \qquad \text{(A.2)}$$

where,

$$\varpi_t(\chi_t) \equiv \frac{\tilde{\omega}_t(\chi_t)}{\breve{\omega}_t(\chi_t)} = \frac{1+o\Delta_t(\chi_t)}{1+\xi\Delta_t(\chi_t)} > 1$$

The last result follows from the facts: (i) $\tilde{\omega}_t(\chi_t) = 1$ and $\breve{\omega}_t(\chi_t) = 1$, in the prototype economy; and $\tilde{\omega}_t(\chi_t) = \frac{1+o\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)} > 1$ and (ii) $\breve{\omega}_t(\chi_t) = \frac{1+\xi\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)} < 1$, given **[R1]** and **[R2]**, in the detailed economy. Facts (i) and (ii) combined imply (III.13).

Now, multiplying both hand sides of (A.2) by $\beta^t\frac{u_t^h}{c_t}$ and summing up from period 0 to any finite period T, gives:

$$\sum_{t=0}^{T}\beta^t u_t^h\left[1-\varpi_t(\chi_t)\frac{(1-\vartheta)h_t}{\vartheta(1-h_t)}\right]=\sum_{t=0}^{T}\beta^t\frac{u_t^h}{c_t}\left\langle-(1+\eta)k_{t+1}+\left\{1+(1-\tau_t^K)\left[\alpha\hat{\omega}_t(\chi_t)\left(\frac{y_t}{k_t}\right)-\delta\right]\right\}k_t\right\rangle$$

Further, note that:

$$\sum_{t=0}^{T}\beta^t\frac{u_t^h}{c_t}\left\langle-(1+\eta)k_{t+1}+\left\{1+(1-\tau_t^K)\left[\alpha\hat{\omega}_t(\chi_t)\left(\frac{y_t}{k_t}\right)-\delta\right]\right\}k_t\right\rangle=$$

$$\frac{u_0^h}{c_0}\left\{1+(1-\tau_0^K)\left[\alpha\hat{\omega}_0(\chi_0)\left(\frac{y_0}{k_0}\right)-\delta\right]\right\}k_0-$$

$$\sum_{t=0}^{T-1}\beta^t\left\langle\frac{u_t^h}{c_t}(1+\eta)-\beta\frac{u_{t+1}^h}{c_{t+1}}\left\{1+(1-\tau_{t+1}^K)\left[\alpha\hat{\omega}_{t+1}(\chi_{t+1})\left(\frac{y_{t+1}}{k_{t+1}}\right)-\delta\right]\right\}\right\rangle k_{t+1}-$$

$$\beta^T\frac{u_T^h}{c_T}(1+\eta)k_{T+1}$$

By taking expectations, based on the information available at the beginning of period 0, applying the Iterated Expectations Theorem in view of the fact that the information of economic agents about the state of the economy is increasing, using the fact that c_t, h_t, k_{t+1} are chosen by economic agents as functions of the information available at the beginning of period t, z^t, and taking the limit as $T\to\infty$, the preceding equation gives:

$$E_0\sum_{t=0}^{\infty}\beta^t\frac{u_t^h}{c_t}\left\langle-(1+\eta)k_{t+1}+\left\{1+(1-\tau_t^K)\left[\alpha\hat{\omega}_t(\chi_t)\left(\frac{y_t}{k_t}\right)-\delta\right]\right\}k_t\right\rangle=$$

$$\frac{u_0^h}{c_0}\left\{1+(1-\tau_0^K)\left[\alpha\hat{\omega}_0(\chi_0)\left(\frac{y_0}{k_0}\right)-\delta\right]\right\}k_0-$$

$$E_0\sum_{t=0}^{\infty}\beta^t\left\langle\frac{u_t^h}{c_t}(1+\eta)-\beta E_t\frac{u_{t+1}^h}{c_{t+1}}\left\{1+(1-\tau_{t+1}^K)\left[\alpha\hat{\omega}_{t+1}(\chi_{t+1})\left(\frac{y_{t+1}}{k_{t+1}}\right)-\delta\right]\right\}\right\rangle k_{t+1}-$$

$$\lim_{T\to\infty}\beta^T E_0\frac{u_T^h}{c_T}(1+\eta)k_{T+1}$$

Then, it follows from the household Euler condition for capital (III.5) that the second term in the RHS of the above equation is zero. Moreover, it follows from the Transversality condition (III.6) that the fourth term of the above expression is zero. Therefore, it follows from (A.2), that along any PSE, we must have:

$$E_0\sum_{t=0}^{\infty}\beta^t u_t^h\left[1-\varpi_t(\chi_t)\frac{(1-\vartheta)h_t}{\vartheta(1-h_t)}\right]=\frac{u_0^h}{c_0}\left\{1+(1-\tau_0^K)\left[\alpha\hat{\omega}_0(\chi_0)\left(\frac{y_0}{k_0}\right)-\delta\right]\right\}k_0 \qquad \text{(A.3)}$$

Clearly, then, in view of definitions (H.2) and (H.3), (A.3) implies the IC (H.1).

Collecting results, the resource constraint (III.2) and the IC are equivalent to all other conditions characterizing the aggregate state of the PSE, except the Initial condition, (III.7). That is, the IC incorporates the government budget constraint (III.3), the intra-temporal condition (III.4), the Euler condition (III.5) and the Transversality condition (III.6).

Finally, given the government spending policy function, $\Delta_t^g(\chi_t,\chi_{t+1})$, defined in (III.15), the aggregate production function, (III.1) and the resource constraint, (III.2), imply the GSPARC, (H.5).

Along any PSE, the labor tax rate is set according to the "government spending policy augmented government budget constraint" (GSPAGBC):

$$\tau_t^L=\frac{\Delta_t^g(\chi_t,\chi_{t+1})-\tau_t^K\left[\alpha\hat{\omega}_t(\chi_t)-\dfrac{\delta}{\bar{\omega}_t(\chi_t)}\left(\dfrac{k_t}{h_t}\right)^{(1-\alpha)}\right]}{(1-\alpha)\tilde{\omega}_t(\chi_t)} \qquad \text{(H.6)}$$

This simply follows from the government budget constraint (III.3), in view of the facts that: $\frac{g_t}{y_t}=\Delta_t^g(\chi_t,\chi_{t+1})$ and $\frac{k_t}{y_t}=\frac{1}{\bar{\omega}_t(\chi_t)}\left(\frac{k_t}{h_t}\right)^{(1-\alpha)}$.

B. Preferences of Insiders

(a) Given the PSE representation it follows that:

$$U_0^i = E_0 \sum_{t=0}^{\infty} \beta^t v^i \left[h_t(z^t), k_t(z^{t-1}), \chi_t(z^{t-1}); z_t \right] \tag{H.7}$$

where:

$$\mathrm{v}^i \left[h_t(z^t), k_t(z^{t-1}), \chi_t(z^{t-1}); z_t \right] \equiv \mathrm{v}_t^i(h_t, k_t, \chi_t) \equiv \hat{\omega}_t(\chi_t) k_t^{\alpha} h_t^{1-\alpha} \tag{H.8}$$

$$\hat{\omega}_t(\chi_t) \equiv \ddot{\omega}_t(\chi_t) \bar{\omega}_t(\chi_t) \tag{H.9}$$

$$\ddot{\omega}_t(\chi_t) \equiv (1-\alpha\theta) \frac{\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)} \tag{H.10}$$

(b) If $\phi \in \left(-\frac{\theta}{1-\theta}, 0 \right)$ and **[R3]** hold, then: $\hat{\omega}(\bullet) : (0,1) \to (0,1)$;

$\lim_{\chi \to 0} \hat{\omega}(\chi) = 0$, $\lim_{\chi \to 1} \hat{\omega}(\chi) = 1$; and $\hat{\omega}'(\chi) > 0, \forall \chi \in (0,1)$.

Proof

Part a. In view of the definition of the objective function of insiders' government in (III.18) and the symmetry that characterizes the PSE, we have:

$$U_0^i = E_0 \left[\sum_{t=0}^{\infty} \beta^t \chi_t \left(w_t^i h_t^i + \pi_t^i \right) \right] \tag{A.4}$$

Then, note that:

$$\chi_t (w_t^i h_t^i + \pi_t^{i(1-\mu)}) = \chi_t (p_t^i y_t^i - r_t\, k_t^i) = \chi_t (1-\alpha\theta) p_t^i y_t^i$$

where the last equality follows from (I.18).

Moreover, it follows from PSE conditions (I.21), (II.16) and (II.17), that:

$$(1-\alpha\theta)\chi_t p_t^i y_t^i = \ddot{\omega}(\chi_t) y_t = \ddot{\omega}(\chi_t) \bar{\omega}(\chi_t) k_t^{\alpha} h_t^{1-\alpha} = \hat{\omega}(\chi_t) k_t^{\alpha} h_t^{1-\alpha}$$

where:

$$\ddot{\omega}_t(\chi_t) \equiv (1-\alpha\theta) \frac{\Delta_t(\chi_t)}{1+\Delta_t(\chi_t)} \tag{A.5}$$

Part b. Observe that, from Proposition 3 in Kollintzas et al (2021), if $\phi \in \left(-\frac{\theta}{1-\theta}, 0 \right)$ and **[R3]** hold, the following are true:

(i) $\bar{\omega}(\chi) > 0, \forall \chi \in (0,1)$

(ii) $\bar{\omega}(0) = 1$

(iii) $\bar{\omega}(1) = 1$

(iv) $\exists$ *a unique* $\bar{\chi} \in (0,1)$ ∋ $\bar{\omega}'(\chi)\begin{cases} < 0, & \chi < \bar{\chi} \\ = 0, & \chi = \bar{\chi} \\ > 0, & \chi > \bar{\chi} \end{cases}$

Accordingly, if $\phi \in \left(-\frac{\theta}{1-\theta}, 0\right)$ and **[R3]** hold,

(v) $\hat{\omega}(\chi) > 0, \forall \chi \in (0,1)$

(vi) $\lim_{\chi \to 0} \hat{\omega}(\chi) = 0,$

(vii) $\lim_{\chi \to 1} \hat{\omega}(\chi) = 1,$

(viii) and $\hat{\omega}(\bullet)$ is differentiable, $\forall \chi \in (0,1)$.

Moreover, it follows after some algebra that:

$\hat{\omega}'(\chi) > 0$ if and only if $1 - \frac{1-\phi}{\phi}\chi > -\frac{1-\phi}{\phi}\frac{1}{1+\frac{1}{\Delta(\chi)}} + \alpha\theta\frac{1}{\theta+\frac{1}{\Delta(\chi)}} + (1-\alpha)\xi\frac{1}{\xi+\frac{1}{\Delta(\chi)}}$

Note then that the LHS of the above inequality is a strictly increasing linear function of χ that takes the value $\chi = 0$ and the value $1 - \frac{1-\phi}{\phi}$ for $\chi = 1$. But, it follows from the properties of $\Delta(\bullet)$, established in Proposition 3 in Kollintzas et al (2021); that, for $\phi \in \left(-\frac{\theta}{1-\theta}, 0\right)$, the RHS of the last inequality is a strictly positive, differentiable, and strictly increasing function of χ, that approaches zero as $\chi \to 0$ and $\alpha\theta + (1-\alpha)\xi - \frac{1-\phi}{\phi} < 1 - \frac{1-\phi}{\phi}$ as $\chi \to 1$. Therefore, the above inequality is always satisfied for all $\chi \in (0,1)$. Therefore, it follows that:

(ix) $\hat{\omega}'(\chi) > 0, \forall \chi \in (0,1)$.

C. The Problem of the Hybrid Government – Primal Approach formulation

Given the vector of stochastic exogenous variables $\{z^t\}_{t=0}^{\infty}$, a politico-economic equilibrium in the case of a Hybrid Government is a sequence of the form $\{h_t^*(z^t), k_{t+1}^*(z^t), \chi_{t+1}^*(z^t)\}_{t=0}^{\infty}$, such that:

$$\{h_t^*(z^t), k_{t+1}^*(z^t), \chi_{t+1}^*(z^t)\}_{t=0}^{\infty} = \underset{\{h_t(z^t), k_{t+1}(z^t), \chi_{t+1}(z^t)\}_{t=0}^{\infty} \in \Im_0}{\arg\max} \; U_0^{\rho}$$

$\Im_0$ is the space of sequences of the form $\left\{h_t(z^t), k_{t+1}(z^t), \chi_{t+1}(z^t)\right\}_{t=0}^{\infty}$ such that:

The IC (H.1), the GSPARC (H.6), and the initial condition (III.7) are satisfied; and, (b) $\left(h_t(z^t), k_{t+1}(z^t), \chi_{t+1}(z^t)\right) \in (0,1)\times(0,\infty)\times(0,1), \forall t \geq 0.$

C1. First Order Conditions for $t = 0$

$$\boldsymbol{M}_{h_0}^{\rho} + \boldsymbol{N}_{h_0} + (1-\alpha)(1-\Delta_0^g)\bar{\omega}_t\left(\frac{k_0}{h_0}\right)^{\alpha}(\boldsymbol{M}_{c_0}^{\rho} + \boldsymbol{N}_{c_0}) = 0 \tag{A.6}$$

$$(1+\eta)(\boldsymbol{M}_{c_0}^{\rho} + \boldsymbol{N}_{c_0}^{\rho}) = \beta E_0\left\{\boldsymbol{M}_{k_1}^{\rho} + \boldsymbol{M}_{c_1}^{\rho}\left[(1-\delta) + \alpha(1-\Delta_1^g)\bar{\omega}_1\left(\frac{k_1}{h_1}\right)^{\alpha-1}\right]\right\} \tag{A.7}$$

$$(\boldsymbol{M}_{c_0}^{\rho} + \boldsymbol{N}_{c_0}^{\rho})\frac{\partial\Delta_0^g}{\partial\chi_1}\bar{\omega}_0 k_0^{\alpha} h_0^{1-\alpha} = \beta E_0\left\{\boldsymbol{M}_{\chi_1}^{\rho} + \boldsymbol{M}_{c_1}^{\rho}\left[-\frac{\partial\Delta_1^g}{\partial\chi_1} + (1-\Delta_1^g)\frac{\bar{\omega}_1'(\chi_1)}{\bar{\omega}_1(\chi_1)}\right]\bar{\omega}_1 k_1^{\alpha} h_1^{1-\alpha}\right\} \tag{A.8}$$

where:

$$\boldsymbol{N}_{c_0} = \frac{\vartheta(1-\gamma)-1}{c_0}\boldsymbol{m}\left(\frac{\hat{\varsigma}_0}{c_0}\right)\left(\frac{u_0^h}{\bar{U}_0^h}\right) \tag{A.9}$$

$$\boldsymbol{N}_{h_0} = \left[-\frac{(1-\vartheta)(1-\gamma)}{1-h_0} + \frac{\frac{\partial\hat{\varsigma}_0}{\partial h_0}\frac{h_0}{\hat{\varsigma}_0}}{h_0}\right]\boldsymbol{m}\left(\frac{\hat{\varsigma}_0}{c_0}\right)\left(\frac{u_0^h}{\bar{U}_0^h}\right) \tag{A.10}$$

C2. First Order Conditions for $t > 0$

Let $\boldsymbol{m}$ be the multiplier associated with the IC and let:

$$\boldsymbol{M}^{\rho}(c_t, h_t, k_t, \chi_t; z_t) \equiv \left[1-\rho-\boldsymbol{m}\tilde{\varsigma}(h_t, \chi_t; z_t)\right]\left[\frac{u^h(c_t, h_t)}{\bar{U}_0^h}\right] + \rho\left[\frac{v^i(h_t, k_t, \chi_t; z_t)}{\bar{U}_0^i}\right] \tag{A.11}$$

where, c_t is given by the GSPARC (H.5). Clearly, if $\{h_t(z^t), k_{t+1}(z^t), \chi_{t+1}(z^t)\}_{t=0}^{\infty}$ is a politico-economic equilibrium in the case of a Hybrid government, then the IC (H.1) and the GSPARC (H.5) must be satisfied along with the following first order conditions, for all $t>0$:

$$\boldsymbol{M}_{h_t}^{\rho} + (1-\alpha)(1-\Delta_t^g)\bar{\omega}_t\left(\frac{k_t}{h_t}\right)^{\alpha}\boldsymbol{M}_{c_t}^{\rho} = 0 \tag{A.12}$$

$$(1+\eta)\boldsymbol{M}_{c_t}^{\rho} = \beta E_t\left\{\boldsymbol{M}_{k_{t+1}}^{\rho} + \boldsymbol{M}_{c_{t+1}}^{\rho}\left[(1-\delta)+\alpha(1-\Delta_{t+1}^g)\bar{\omega}_{t+1}\left(\frac{k_{t+1}}{h_{t+1}}\right)^{\alpha-1}\right]\right\} \tag{A.13}$$

$$\boldsymbol{M}_{c_t}^{\rho}\frac{\partial\Delta_t^g}{\partial\chi_{t+1}}\bar{\omega}_t k_t^{\alpha}h_t^{1-\alpha} = \beta E_t\left\{\boldsymbol{M}_{\chi_{t+1}}^{\rho} + \boldsymbol{M}_{c_{t+1}}^{\rho}\left[-\frac{\partial\Delta_{t+1}^g}{\partial\chi_{t+1}} + (1-\Delta_{t+1}^g)\frac{\bar{\omega}_{t+1}'(\chi_{t+1})}{\bar{\omega}_{t+1}(\chi_{t+1})}\right]\bar{\omega}_{t+1}k_{t+1}^{\alpha}h_{t+1}^{1-\alpha}\right\} \tag{A.14}$$

where:

$$\boldsymbol{M}_{c_t}^{\rho} = \frac{\vartheta(1-\gamma)(1-\rho-\boldsymbol{m}\tilde{\varsigma}_t)\left(\frac{u_t^h}{\bar{U}_0^h}\right)}{c_t} \tag{A.15}$$

$$\boldsymbol{M}_{h_t}^{\rho} = -\frac{(1-\vartheta)(1-\gamma)(1-\rho-\boldsymbol{m}\tilde{\varsigma}_t)\left(\frac{u_t^h}{\bar{U}_0^h}\right)}{1-h_t} - \frac{\boldsymbol{m}\tilde{\varsigma}_t\left(\frac{\partial\tilde{\varsigma}_t}{\partial h_t}\frac{h_t}{\tilde{\varsigma}_t}\right)\left(\frac{u_t^h}{\bar{U}_0^h}\right)}{h_t} + (1-\alpha)\rho\left(\frac{v_t^i}{\bar{U}_0^i}\right)\left(\frac{1}{h_t^i}\right) =$$

$$-\frac{(1-\vartheta)(1-\gamma)(1-\rho-\boldsymbol{m}\tilde{\varsigma}_t)\left(\frac{u_t^h}{\bar{U}_0^h}\right)}{1-h_t} - \frac{\boldsymbol{m}\tilde{\varsigma}_t\left(\frac{\partial\tilde{\varsigma}_t}{\partial h_t}\frac{h_t}{\tilde{\varsigma}_t}\right)\left(\frac{u_t^h}{\bar{U}_0^h}\right)}{h_t} + (1-\alpha)\left(\frac{\rho}{\bar{U}_0^i}\right)\hat{\omega}_t(\chi_t)\bar{\omega}_t(\chi_t)\left(\frac{k_t}{h_t}\right)^{\alpha} \tag{A.16}$$

$$\boldsymbol{M}_{k_{t+1}}^{\rho} = \alpha\rho\left(\frac{v_{t+1}^i}{\bar{U}_0^i}\right)\left(\frac{1}{k_{t+1}^i}\right) = \alpha\left(\frac{\rho}{\bar{U}_0^i}\right)\hat{\omega}_{t+1}(\chi_{t+1})\bar{\omega}_{t+1}(\chi_{t+1})\left(\frac{k_{t+1}}{h_{t+1}}\right)^{\alpha-1} \tag{A.17}$$

$$\begin{aligned}\boldsymbol{M}_{\chi_{t+1}}^{\rho} &= -\boldsymbol{m}\tilde{\varsigma}_{t+1}\left(\frac{\partial\tilde{\varsigma}_{t+1}}{\partial\chi_{t+1}}\frac{1}{\tilde{\varsigma}_{t+1}}\right)\left(\frac{u_{t+1}^h}{\bar{U}_0^h}\right) + \rho\left(\frac{\hat{\omega}_{t+1}'(\chi_{t+1})}{\hat{\omega}_{t+1}(\chi_{t+1})}\right)\left(\frac{v_{t+1}^i}{\bar{U}_0^i}\right)\\ &= -\boldsymbol{m}\tilde{\varsigma}_{t+1}\left(\frac{\partial\tilde{\varsigma}_{t+1}}{\partial\chi_{t+1}}\frac{1}{\tilde{\varsigma}_{t+1}}\right)\left(\frac{u_{t+1}^h}{\bar{U}_0^h}\right) + \left(\frac{\rho}{\bar{U}_0^i}\right)\left(\frac{\hat{\omega}_{t+1}'(\chi_{t+1})}{\hat{\omega}_{t+1}(\chi_{t+1})}\right)\hat{\omega}_{t+1}(\chi_{t+1})\bar{\omega}_{t+1}(\chi_{t+1})k_{t+1}^{\alpha}h_{t+1}^{1-\alpha}\end{aligned} \tag{A.18}$$

C3. The Asymptotic Steady State:

We consider the asymptotic steady state $(c,h,k,\chi,\boldsymbol{m})$ implied by the first-order conditions for $t>0$, (A.12)-(A.14), the IC (H.1) for $(k_0,\chi_0)=(k,\chi)$, and the GSPARC (H.5).

In order to condense notation, we also use steady state output, $y = \bar{\varpi}(\chi)k^{\alpha}h^{(1-\alpha)}$; and, steady state output share of government expenditures in the detailed economy, $\Delta^g = \bar{\psi} + \hat{\psi}\chi$. Then, the asymptotic steady state of the PE in the case of a Hybrid government is characterized by the following equations:

$$\left(\frac{-\boldsymbol{M}_h^{\rho}}{\boldsymbol{M}_c^{\rho}}\right)\left(\frac{h}{y}\right) = \left[\frac{(1-\vartheta)\left(\frac{c}{y}\right)h}{\vartheta} - \frac{\boldsymbol{m}(1-\tilde{\varsigma})\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)(1-\rho-\boldsymbol{m}\tilde{\varsigma})}\right]\left(\frac{1}{1-h}\right)$$
$$+(1-\alpha)\frac{\rho\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right)\left(\frac{\bar{U}^h}{u^h}\right)\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)(1-\rho-\boldsymbol{m}\tilde{\varsigma})} = (1-\alpha)(1-\Delta^g) \tag{A.19}$$

$$\left(\frac{k}{y}\right) = \frac{\alpha(1-\Delta^g)+\left(\frac{\boldsymbol{M}_k^{\rho}}{\boldsymbol{M}_c^{\rho}}\right)\left(\frac{k}{y}\right)}{\beta^{-1}(1+\eta)-(1-\delta)} = \frac{\alpha\left[(1-\Delta^g)+\frac{\rho\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right)\left(\frac{\bar{U}^h}{u^h}\right)\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)(1-\rho-\boldsymbol{m}\tilde{\varsigma})}\right]}{\beta^{-1}(1+\eta)-(1-\delta)} \tag{A.20}$$

$$\left(\frac{\boldsymbol{M}_{\chi}^{\rho}}{\boldsymbol{M}_c^{\rho}}\right)\left(\frac{1}{y}\right) = \frac{\boldsymbol{m}(1-\tilde{\varsigma})\left(\frac{c}{y}\right)+}{\vartheta(1-\gamma)(1-\rho-\boldsymbol{m}\tilde{\varsigma})}\left(\frac{\varpi'(\chi)}{\varpi(\chi)}\right) + \frac{\rho\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right)\left(\frac{\bar{U}^h}{u^h}\right)\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)(1-\rho-\boldsymbol{m}\tilde{\varsigma})}\left(\frac{\hat{\omega}'(\chi)}{\hat{\omega}(\chi)}\right) = \tag{A.21}$$
$$= \frac{\partial\Delta^g}{\partial\chi} - (1-\Delta^g)\frac{\bar{\varpi}'(\chi)}{\bar{\varpi}(\chi)}$$

C4. Great Ratios and <u>Auxiliary Variables</u> in the the Asymptotic Steady State

The asymptotic steady state of the politico-economic equilibrium can be characterized in terms of the four great ratios $\left[\left(\frac{c}{y}\right), h, \left(\frac{k}{y}\right), \chi\right]$. And, moreover, $\left(\frac{c}{y}\right)$, h and $\left(\frac{k}{y}\right)$ can be expressed, analytically, in terms of χ.

In the five-equation system (H.1), (H.5) and (A.19) – (A.21), the multiplier $\boldsymbol{m}$ can be eliminated, so that the asymptotic steady state of the politico-economic equilibrium can be characterized in terms of the four great ratios $\left[\left(\frac{c}{y}\right), h, \left(\frac{k}{y}\right), \chi\right]$. And, moreover, $\left(\frac{k}{y}\right)$, $\left(\frac{c}{y}\right)$, and h can be expressed, analytically, in terms of χ.

Suppose that $\rho \in (0,1)$. Then, $(c,h,k,\chi,\boldsymbol{m})$ is an asymptotic steady state of the politico-economic equilibrium with a hybrid government, if and only if, the four great ratios: $\left(\frac{k}{y}\right), \left(\frac{c}{y}\right), h, \chi$ and the three auxiliary variables: $\tilde{\varsigma}, \bar{\varsigma}, \hat{\varsigma}$ constitute a solution to the following seven-equation system:

$$\left(\frac{c}{y}\right) = (1-\bar{\psi}-\hat{\psi}\chi) - (\eta+\delta)\left(\frac{k}{y}\right) \tag{A.22}$$

$$\tilde{\varsigma} = \frac{(1-\beta)\left\{a(1-\tau^K)\hat{\omega}(\chi) + [1-(1-\tau^K)\delta]\left(\frac{k}{y}\right)\right\}}{\left(\frac{c}{y}\right)} \tag{A.23}$$

$$h = \frac{1}{1+\frac{(1-\vartheta)\varpi(\chi)}{\vartheta(1-\tilde{\varsigma})}} \tag{A.24}$$

$$\bar{\varsigma} = \left(\frac{1-\vartheta}{\vartheta}\right)\left(\frac{c}{y}\right)h - (1-\alpha)(1-\bar{\psi}-\hat{\psi}\chi-\hat{\varsigma})(1-h) \tag{A.25}$$

$$\left(\frac{k}{y}\right) = \frac{\alpha(1-\bar{\psi}-\hat{\psi}\chi+\hat{\varsigma})}{\beta^{-1}(1+\eta)-(1-\delta)} \tag{A.26}$$

$$\bar{\varsigma}\left(\frac{\varpi'(\chi)}{\varpi(\chi)}\right) + \hat{\varsigma}\left(\frac{\hat{\omega}'(\chi)}{\hat{\omega}(\chi)}\right) = \hat{\psi} - \frac{\bar{\omega}'(\chi)}{\bar{\omega}(\chi)}(1-\bar{\psi}-\hat{\psi}\chi) \tag{A.27}$$

$$\hat{\varsigma} = \varepsilon^{\rho}\left[\frac{(1-\tilde{\varsigma})\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)\tilde{\varsigma}} + \bar{\varsigma}\right] \tag{A.28}$$

where:

$$\varepsilon^{\rho} \equiv \left[\rho\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right) \Big/ (1-\rho)\left(\frac{u^h}{\bar{U}^h}\right)\right] = \left[\rho\left(\frac{\mathrm{v}^i}{\bar{\mathrm{v}}^i}\right) \Big/ (1-\rho)\left(\frac{u^h}{\bar{u}^h}\right)\right] = -\frac{dU^h}{dU^i}\frac{U^i}{U^h}\Big|_{dU^{\rho}=0} \tag{A.29}$$

Proof

By definition, $(c,h,k,\chi,\boldsymbol{m})$ is an asymptotic steady state of the politico-economic equilibrium with a hybrid government, if and only if, it is a solution to the following five-equation system: (A.19) – (A.21) and the GSPARC, (H.5) and the IC, (H.1), evaluated at $(c,h,k,\chi,\boldsymbol{m})$.

Observe, first that (H.1) and (H.5) can be equivalently written:

$$\left(\frac{c}{y}\right)=(1-\bar{\psi}+\hat{\psi}\chi)-(\eta+\delta)\left(\frac{k}{y}\right) \tag{A.30}$$

and

$$\tilde{\varsigma}=\frac{(1-\beta)\left\{a(1-\tau^{K})\bar{\omega}(\chi)+[1-(1-\tau^{K})\delta]\left(\frac{k}{y}\right)\right\}}{\left(\frac{c}{y}\right)} \tag{A.31}$$

Here, in going from (H.1) and (H.5) to (A.30) and (A.31), respectively, we have used the definitions:

$$\Delta^{g}=\bar{\psi}+\hat{\psi}\chi \tag{A.32}$$

$$y=\bar{\omega}(\chi)k^{\alpha}h^{1-\alpha} \tag{A.33}$$

$$\tilde{\varsigma}\equiv[1-\varpi(\chi)\frac{(1-\vartheta)h}{\vartheta(1-h)}] \tag{A.34}$$

It is apparent that (A.30) is (III.2), (A.31) is (A.23) and (A.34) yields (A.24).

Then, note that, in view of (A.32) – (A.34), conditions (A.19) – (A.21) can be re-written as follows:

$$\left(\frac{1}{1-h}\right)\left[\frac{(1-\vartheta)h}{\vartheta}-\frac{\boldsymbol{m}(1-\tilde{\varsigma})}{\vartheta(1-\gamma)\left(1-\rho-\boldsymbol{m}\tilde{\varsigma}\right)}\right]\left(\frac{c}{y}\right)+(1-\alpha)\frac{\rho\left(\frac{\mathrm{v}^{i}}{\bar{U}^{i}}\right)\left(\frac{\bar{U}^{h}}{u^{h}}\right)\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)\left(1-\rho-\boldsymbol{m}\tilde{\varsigma}\right)}=(1-\alpha)(1-\bar{\psi}-\hat{\psi}\chi) \tag{A.35}$$

$$\left(\frac{k}{y}\right)=\frac{\alpha\left[(1-\bar{\psi}-\hat{\psi}\chi)+\frac{\rho\left(\frac{\mathrm{v}^{i}}{\bar{U}^{i}}\right)\left(\frac{\bar{U}^{h}}{u^{h}}\right)\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)(1-\rho-\boldsymbol{m}\tilde{\varsigma})}\right]}{\beta^{-1}(1+\eta)-(1-\delta)} \tag{A.36}$$

$$\frac{\boldsymbol{m}(1-\tilde{\varsigma})\left(\frac{\varpi'(\chi)}{\varpi(\chi)}\right)+\rho\left(\frac{\mathrm{v}^{i}}{\bar{U}^{i}}\right)\left(\frac{\bar{U}^{h}}{u^{h}}\right)\left(\frac{\hat{\omega}'(\chi)}{\hat{\omega}(\chi)}\right)}{\vartheta(1-\gamma)\left(1-\rho-\boldsymbol{m}\tilde{\varsigma}\right)}\left(\frac{c}{y}\right)=\hat{\psi}-\frac{\bar{\omega}'(\chi)}{\bar{\omega}(\chi)}(1-\bar{\psi}-\hat{\psi}\chi) \tag{A.37}$$

Now, since by assumption $\rho \in (0,1)$, observe that:

$$\frac{\boldsymbol{m}(1-\tilde{\varsigma})}{\vartheta(1-\gamma)\left(1-\rho-\boldsymbol{m}\tilde{\varsigma}\right)} = \frac{\left(\frac{\boldsymbol{m}}{1-\rho}\right)(1-\tilde{\varsigma})}{\vartheta(1-\gamma)\left[1-\left(\frac{\boldsymbol{m}}{1-\rho}\right)\tilde{\varsigma}\right]} \tag{A.38}$$

and

$$\frac{\rho\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right)\left(\frac{\bar{U}^h}{u^h}\right)}{\vartheta(1-\gamma)\left(1-\rho-\boldsymbol{m}\tilde{\varsigma}\right)} = \frac{\left(\frac{\rho}{1-\rho}\right)\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right)\left(\frac{\bar{U}^h}{u^h}\right)}{\vartheta(1-\gamma)\left[1-\left(\frac{\boldsymbol{m}}{1-\rho}\right)\tilde{\varsigma}\right]} \tag{A.39}$$

However, since $\boldsymbol{m} \neq 0$ is an unknown real number, so is $\left(\frac{\boldsymbol{m}}{1-\rho}\right)$. For that matter, without loss of generality, we can re-write (A.35) - (A.37) as follows:

$$\left(\frac{1}{1-h}\right)\left[\frac{(1-\vartheta)h}{\vartheta} - \frac{\boldsymbol{m}(1-\tilde{\varsigma})}{\vartheta(1-\gamma)\left(1-\boldsymbol{m}\tilde{\varsigma}\right)}\right]\left(\frac{c}{y}\right) + (1-\alpha)\frac{\left(\frac{\rho}{1-\rho}\right)\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right)\left(\frac{\bar{U}^h}{u^h}\right)\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)\left(1-\boldsymbol{m}\tilde{\varsigma}\right)} = (1-\alpha)(1-\bar{\psi}-\hat{\psi}\chi) \tag{A.40}$$

$$\left(\frac{k}{y}\right) = \frac{\alpha\left[(1-\bar{\psi}-\hat{\psi}\chi) + \frac{\left(\frac{\rho}{1-\rho}\right)\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right)\left(\frac{\bar{U}^h}{u^h}\right)\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)(1-\boldsymbol{m}\tilde{\varsigma})}\right]}{\beta^{-1}(1+\eta)-(1-\delta)} \tag{A.41}$$

$$\frac{\boldsymbol{m}(1-\tilde{\varsigma})\left(\frac{\varpi'(\chi)}{\varpi(\chi)}\right) + \left(\frac{\rho}{1-\rho}\right)\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right)\left(\frac{\bar{U}^h}{u^h}\right)\left(\frac{\hat{\omega}'(\chi)}{\hat{\omega}(\chi)}\right)}{\vartheta(1-\gamma)\left(1-\boldsymbol{m}\tilde{\varsigma}\right)}\left(\frac{c}{y}\right) = \hat{\psi} - \frac{\bar{\omega}'(\chi)}{\bar{\omega}(\chi)}(1-\bar{\psi}-\hat{\psi}\chi) \tag{A.42}$$

Clearly then, if we set:

$$\bar{\varsigma} \equiv \frac{\boldsymbol{m}(1-\tilde{\varsigma})}{\vartheta(1-\gamma)\left(1-\boldsymbol{m}\tilde{\varsigma}\right)}\left(\frac{c}{y}\right) \tag{A.43}$$

$$\hat{\varsigma} \equiv \frac{\left(\frac{\rho}{1-\rho}\right)\left(\frac{\mathrm{v}^i}{\bar{U}^i}\right)\left(\frac{\bar{U}^h}{u^h}\right)}{\vartheta(1-\gamma)\left(1-\boldsymbol{m}\tilde{\varsigma}\right)}\left(\frac{c}{y}\right) = \frac{\varepsilon^{\rho}}{\vartheta(1-\gamma)\left(1-\boldsymbol{m}\tilde{\varsigma}\right)}\left(\frac{c}{y}\right) \tag{A.44}$$

where, the last equality in (A.44) follows from the definition of ε^{ρ} in (A.29). Then, it is straightforward that, in view of definitions (A.43) and (A.44): (A.40) yields (A.25).

Finally, observe that the auxiliary variables $\tilde{\varsigma}, \hat{\varsigma}, \overline{\varsigma}$ are not independent of each other and their very definitions imply that they should satisfy (A.28), as a consistency requirement. To see this note that (A.43) and (A.44) can be solved for the multiplier, $\boldsymbol{m}$, to give:

$$\boldsymbol{m} = \frac{\vartheta(1-\gamma)\overline{\varsigma}}{(1-\tilde{\varsigma})\left(\frac{c}{y}\right)+\vartheta(1-\gamma)\tilde{\varsigma}\overline{\varsigma}} = \frac{\vartheta(1-\gamma)\hat{\varsigma}-\varepsilon^{\rho}\left(\frac{c}{y}\right)}{\vartheta(1-\gamma)\tilde{\varsigma}\hat{\varsigma}} \qquad \text{(A.45)}$$

Then, it is straightforward to show that the second equality in (A.45) yields (A.28).