Desirability

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The two sides of decision making



Heads

Tails

Unobservable

probability

The two sides of decision making

Tails

Heads

Observable

preferred choices

Bets

The two sides of decision making



Heads

Tails

Unobservable

probability

qualitative probability

E is at least as probable as F

unobservable

observable





Pre-Savage

on consequences

probability

qualitative probability

relation on events

Bets

Savage

probability-utility

desirability

relation on events

Acts

Qualitative probability

(Ω, Σ) - measurable space

$\geq \subseteq \Sigma^2$ - Qualitative probability relation

$E \geq F$ *E is at least as probable as F*

1. $\geq$ is complete and reflexive;
2. If $H \cap (E \cup F) = \emptyset$ then $E \geq F \Leftrightarrow E \cup H \geq F \cup H$;
3. $E \geq \emptyset$, $\emptyset \not\geq \Omega$.

Probability $p: \Sigma \to R$ *represents* $\geq$ when
$E \geq F \Leftrightarrow p(E) \geq p(F)$

Theorem (Savage 54)

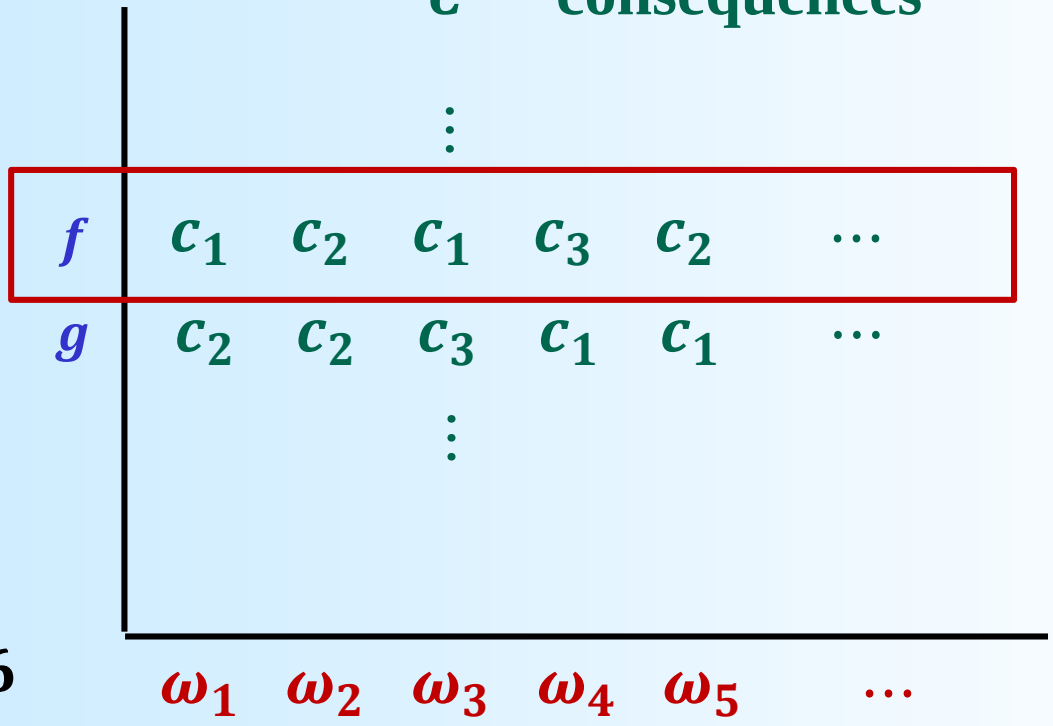
Satisfies 1.-3. and P6' if and only if it is represented by a nonatomic probability p.

Theorem: There exist $p \in \Delta(\Omega)$ $u: C \to R$ such that

$$f \gtrsim g \iff \int u(f(\omega))dp(\omega) \geq \int u(g(\omega))dp(\omega)$$

A
Actions

C – **consequences**



Ω – **states of the world**

Ex ante

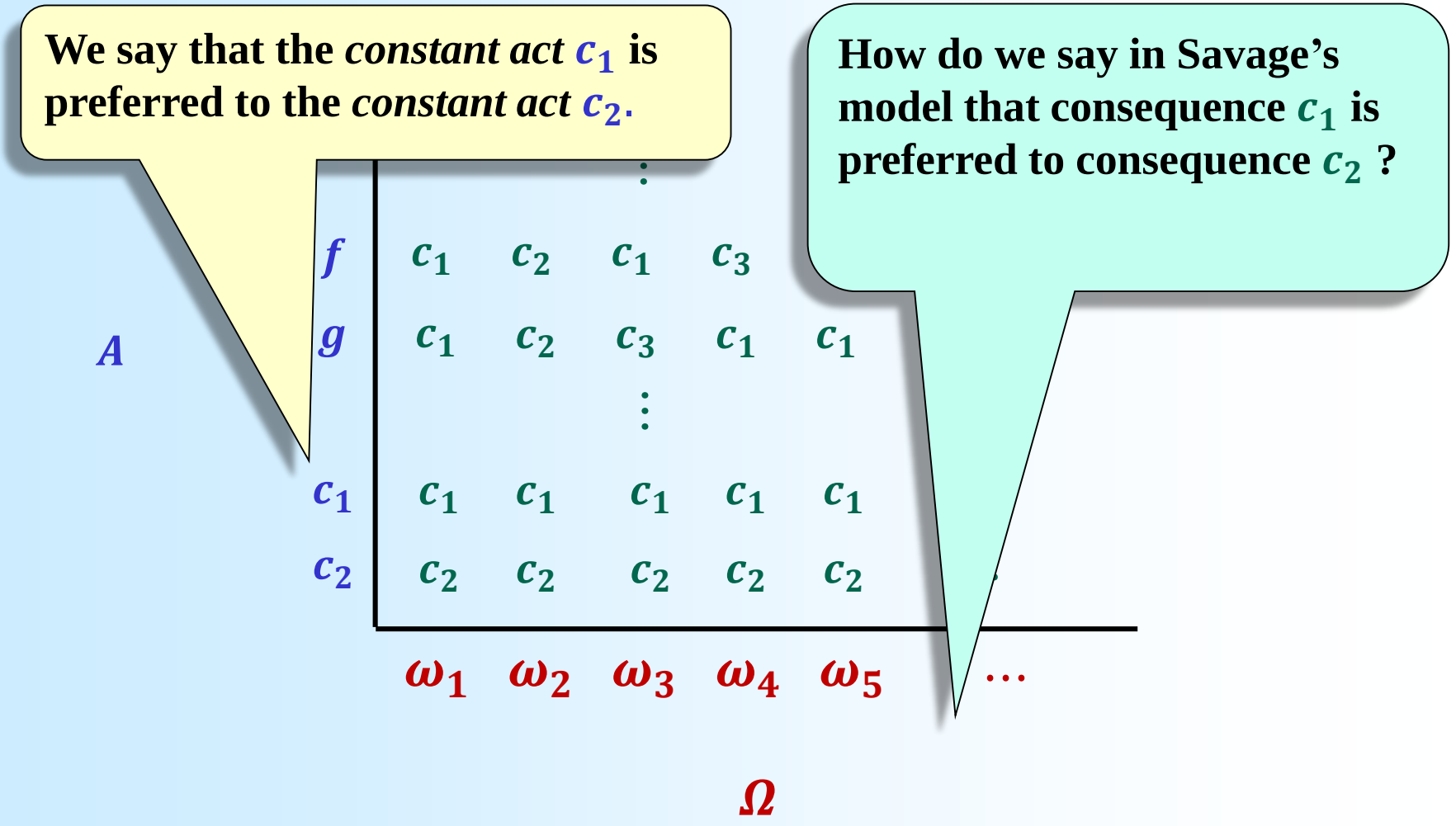
Interim

Comprehensive state space

Replacing (p, u) by a relation on events??

$\gtrsim \subseteq A^2$

Satisfies $P1, \ldots, P6$



Savage's model **Ex ante**

We say that the *event* c_1 is preferred to the *event* c_2.

Preference of events - desirability

f c_1 c_1 c_2 c_2 c_3 $\cdots$ Interim

Comprehensive state space

ω_1 ω_2 ω_3 ω_4 ω_5 $\cdots$

c_1 c_2

How do we say in a comprehensive state space that consequence c_1 is preferred to consequence c_2 ?

Formalities

The model

(Ω, Σ) - A state space and a σ-algebra of events

$\mathcal{C} = \{c_1, \ldots, c_n\}$ - Consequences, $n > 1$

$f: \Omega \to \mathcal{C}$ - A measurable act

$C = f^{-1}(c)$ - The event that the consequence is c

Desirability

$\succsim \subseteq \Sigma^2$ - Desirability

$E \succsim F$

$E \sim F := E \succsim F \succsim E$

$E \succ F := F \succsim E$

E is more desirable than F

The set of null events is an ideal:

- closed under union
- the sub events of null are null

N is null if for each E and F such that $E \succsim F$ also

$$E' \succsim F'$$

whenever $E' \Delta E \subseteq N$ and $F' \Delta F \subseteq N$.

Assume that for each c, C is non-null

"Technical" axioms

Axiom 1. (*weak order*)
$\succsim$ is a complete transitive order on the non-null events

Axiom 2. (*non-triviality*)
There are E and F such that $E \succ F$.

Axiom 3. (*nonatomiciy*)
For $E \succ F$ there exists a partition of Ω $P = (P_1, \ldots, P_m)$ such that for each i:
if $E \Delta E' \subseteq P_i$ then $E' \succ F$ and
if $F \Delta F' \subseteq P_i$ then $E \succ F'$.

Axiom 4. (*pairs*)
If E and F are non-null
and for each pair c_i and c_j

$E \cap (C_i \cup C_j)$ and $F \cap (C_i \cup C_j)$ are
either null
or non-null and as desirable,

Then $E \sim F$.

Substantial axioms

Test your desire:

Acceptance letter from *AER* ∩ **Life found on Mars** ≈? **Acceptance letter from *AER***

Axiom 5. (*constant act*)
For each consequence *c*, if *E* is non-null and *E* ⊆ *C* then *E* ~ *C*.

Substantial axioms

Test your desire:

Acceptance letter from *AER* $\succsim$ **Acceptance letter from *TESN***

Acceptance letter from *AER* $\overset{?}{\succsim}$ **Acceptance letter from *AER*** $\cup$ **Acceptance letter from *TESN***

Axiom 6. (*intermediacy*)
If $E \cap F = \emptyset$ **then the following conditions are equivalent:**

$$E \succsim F$$

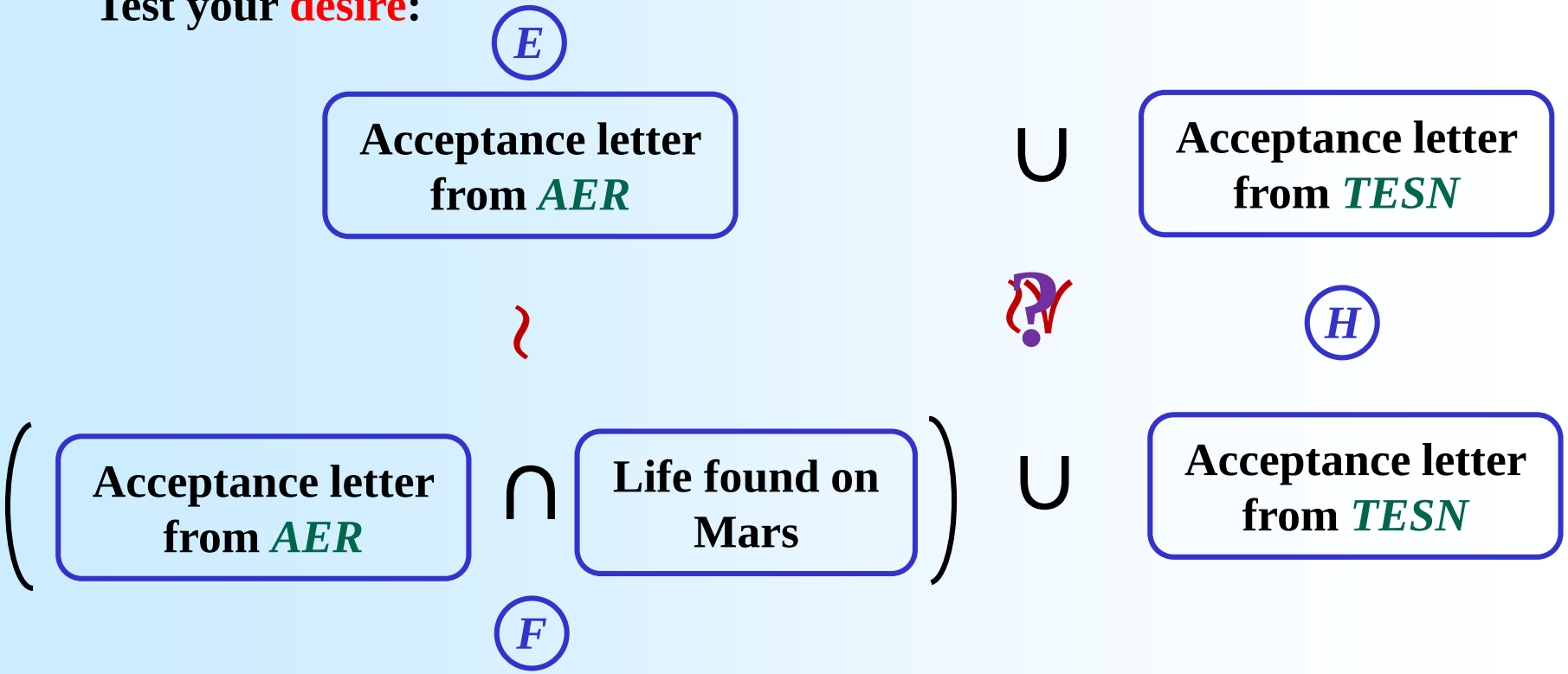
$$E \succsim E \cup F$$

$$E \cup F \succsim F$$

nsactions of the nometric Society f Novosibirsk

Substantial axioms

Test your desire:



$E \sim F \succ H$

$H \cap (E \cup F) = \emptyset$

$E \cup H \gtrsim F \cup H$

notation

$E \gtrsim_H F$

$H \succ E \sim F$

$H \cap (E \cup F) = \emptyset$

$F \cup H \gtrsim E \cup H$

Substantial axioms

Axiom 7. (*persistency*)

If $E \sim F \succ H$ **and** $E \sim F \succ G$

and

$H \cap (E \cup F) = \emptyset$ **and** $G \cap (E \cup F) = \emptyset$ **.**

Then

$E \succsim_H F$ **implies** $E \succsim_G F$**.**

$E \sim F \succ H$

$H \cap (E \cup F) = \emptyset$

$A \cup H \succsim F \cup H$

notation

$E \succsim_H F$

$H \succ E \sim F$

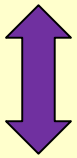
$H \cap (E \cup F) = \emptyset$

$F \cup H \succsim E \cup H$

The expectation of u given E

Representation theorem

$\succsim$ is a desirability relation



there exist a probability $\mu: \Sigma \rightarrow R$

and utility $u: C \rightarrow R$ such that:

the μ-null events are the null events, and

$E \succsim F$ if and only if

$$\Sigma_c \, u(c) \, \mu(C \mid E) \quad \geq \quad \Sigma_c u(c) \mu(C \mid F)$$

The pairs (μ, u) that represent $\succsim$

Assume that $C_n \succ C_{n-1} \succ \cdots \succ C_1$

For each i, the conditionals $\mu(\cdot\,|C_i)$ are uniquely determined.

Let $p = (p_1, \ldots, p_n) = (\mu(C_1), \ldots, \mu(C_n))$

For each p the utility u is uniquely determined up to transformations $\alpha u + \beta(1, \ldots, 1)$ for $\alpha > 0$

The vectors $(u_2 - u_1, u_3 - u_2, \ldots)$ are ordered by LRD in a reverse order of the corresponding p's.

p likelihood ratio dominates (LRD) q ($p \gg q$) if for each i,

$$p_{i+1} / p_i > q_{i+1} / q_i$$

p is *more optimistic* than q.

such that $p^u \gg p^d$ and all p are in the open interval

$$(p^d \text{ ——————— } p^u)$$

$$\uparrow q \qquad p \uparrow$$

if

Then $p \gg q$

Consistent family of desirability relations

$\mathcal{F}$ – a family of acts

$$\mathcal{D} = \{ \succsim^f \mid f \in \mathcal{F} \}$$

Axiom 1. (*joint null events*)
All desirability relations in $\mathcal{D}$ have the same null events.

Axiom 2. (*richness*)
If (H_c) is a partition into non-null events, then there is $g \in \mathcal{F}$ such that for each c, $H_c = g^{-1}(c)$.

Axiom 3. (*joint desirability*)
$f, g \in \mathcal{F}$,

$$f|_A = g|_A \text{ and } f|_B = g|_B$$

$$\Downarrow$$

$$A \succsim^f B \iff A \succsim^g B$$

Axiom 4. (*joint likelihood*)
$f, g \in \mathcal{F}$,

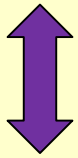
$$\text{for some } H,\ A \succsim^f_H B$$

$$\Updownarrow$$

$$\text{for some } H',\ A \succsim^g_{H'} B$$

Representation theorem

A family of desirability relations $\{\succsim^f\}$ is consistent



there exists a pair (μ, u) that represents each relation $\succsim^f$.

Moreover, the probability μ is uniquely determined and the utility u is uniquely determined up to a positive affine transformation.

Neighbor exchange rate

$\delta = \eta = 1/2$

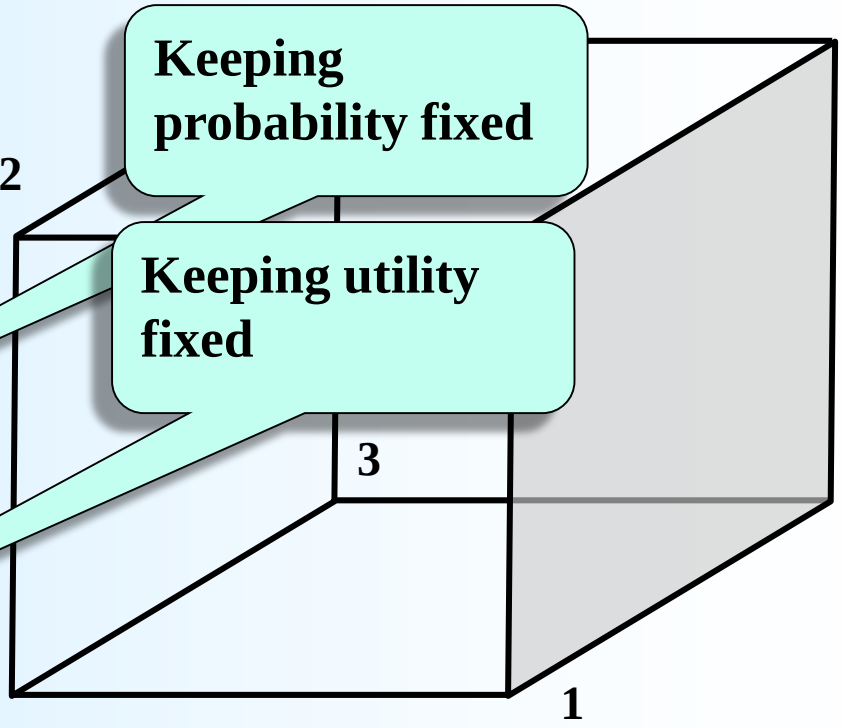
$\uparrow \delta$ $\quad \eta \uparrow$

$\mu(E|C_1)$ $\quad \mu(E|C_2)$ $\quad \mu(E|C_3)$

$\downarrow$ -1

$$\delta\left(\frac{1}{3}\right) + \eta\left(\frac{1}{3}\right) = 1\left(\frac{1}{3}\right)$$

$$\delta\left(\frac{1}{3}\right)0 + \eta\left(\frac{1}{3}\right)1 = 1\left(\frac{1}{3}\right)1/2$$

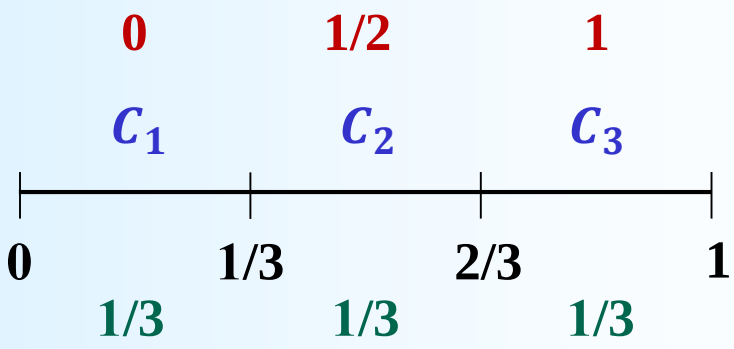


utility

Consequence events

Ω with Lebesgue measure

Prob of consequence



Neighbor exchange rate

$\delta = \eta = 1/2$

$$\left(\frac{1}{2}\right)p_1 + \left(\frac{1}{2}\right)p_3 = 1p_2$$

$(\frac{2}{3}, \frac{1}{3}, 0)$ ——— $(0, \frac{1}{3}, \frac{2}{3})$

p^d p^u

$p = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$

$u = (0, \frac{3}{4}, 1)$

utility

Consequence events

Ω with Lebesgue measure

Prob of consequence

